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Research Article

Controlling the Stochastic Sensitivity in Nonlinear Discrete-Time Systems with Incomplete Information

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For stochastic nonlinear discrete-time system with incomplete information, a problem of the stabilization of equilibrium is considered. Our approach uses a regulator which synthesizes the required stochastic sensitivity. Mathematically, this problem is reduced to the solution of some quadratic matrix equations. A description of attainability sets and algorithms for regulators design is given. The general results are applied to the suppression of unwanted large-amplitude oscillations around the equilibria of the stochastically forced Verhulst model with noisy observations.

1. Introduction

Controlling of the complex systems in nature and society is a challenging and fundamental problem of the modern mathematical theory of nonlinear dynamics and engineering. Discrete dynamic models because of widely used computer-oriented technologies, attract attention of many researchers [1, 2]. Even in simple discrete models, due to nonlinearity, a variety of dynamic regimes, both regular and chaotic, is observed [3–5]. An interplay of nonlinearity and stochasticity can generate new unexpected phenomena [6–10].

A lot of nonlinear systems operate in zones of stochastic transitions from order to chaos. After the pioneering work [11], a problem of controlling chaos is extensively studied [12–14]. Most of the reported results are based on the direct numerical simulation. A detailed theoretical description of stochastic attractor is given by the stationary probabilistic density function. For discrete systems, this function is governed by Frobenius-Perron equation [15]. Unfortunately, an analytical solution of this equation is possible only in very special cases, so, a development of the asymptotic approximations is a highly relevant area of research.

For the constructive analysis of the stochastic attractors of nonlinear discrete-time dynamical systems, a stochastic sensitivity functions technique was elaborated [16]. This technique was applied to the analysis on noise-induced intermittency [17] and neuron excitability [18]. On the base of this technique, a new approach for the solution of control problems in stochastic discrete-time systems was suggested in [19]. In these studies, it was supposed that the complete information about current state of the controlled system is known. However, in many practical situations, the system data are far from complete. For example, only some coordinates of the system state are observable, and, moreover, these observations contain stochastic errors. So, the control of stochastic systems with incomplete information is an urgent research domain [20–22].

In present paper, we further develop a theory for the synthesis of the stochastic sensitivity for the equilibria in a randomly forced control discrete system with incomplete information. Mathematically, presence of noise in the observations leads to a new algebraic analysis of quadratic matrix equations. In Section 2, we introduce the stochastic sensitivity matrix as a basic probabilistic characteristics for the randomly forced equilibria. A problem of the synthesis of this matrix is considered. A important notion of the attainability is discussed here. A problem of the stochastic sensitivity matrix synthesis is reduced to the analysis of the corresponding quadratic matrix equation. Results of this theoretical analysis in the general multidimensional case are presented in a Theorem. This Theorem gives a description of attainability sets and algorithms for regulators design.

One-dimensional case is discussed in details in Section 3. In Section 4, we apply the results to the suppression of unwanted large-amplitude oscillations around the equilibria of the stochastically forced Verhulst model with noisy observations. We show that our regulator can be used for the suppression of chaos.

2. Synthesis of Stochastic Sensitivity

Consider a nonlinear controlled discrete-time stochastic system

$$x_{t+1} = f(x_t, u_t) + \varepsilon\sigma(x_t)\xi_t, \quad (1)$$

where $x, f \in R^n$, $u \in R^l$, $\sigma \in R^{n\times m}$, and u is a control input. Here, $\xi_t \in R^m$ is an uncorrelated random sequence with parameters $\mathrm{E}\xi_t = 0$ and $\mathrm{E}\xi_t\xi_t^\top = I$, and I is the identity $m\times m$-matrix and ε is a scalar parameter of noise intensity.

It is supposed that the corresponding deterministic uncontrolled system (1) (with $\varepsilon = 0$ and $u = 0$ therein) has an equilibrium $\overline{x} : \overline{x} = f(\overline{x}, 0)$. Stability of $\overline{x}$ is not assumed.

In present paper, we consider a case of incomplete information when the measurement vector y_t is known only:

$$y_t = g(x_t) + \varepsilon\varphi(x_t)\eta_t, \quad (2)$$

where $y, g \in R^l$, $\varphi \in R^{l\times k}$. Here, $\eta_t \in R^k$ is an uncorrelated random sequence with parameters $\mathrm{E}\eta_t = 0$ and $\mathrm{E}\eta_t\eta_t^\top = I$, and I is the identity $k\times k$-matrix.

In this circumstance, we consider the following regulator:

$$u_t = K[y_t - g(\overline{x})]. \quad (3)$$

The dynamics of the closed-loop stochastic system (1) with the regulator (3) using noisy observations (2) is governed by the following system:

$$\begin{aligned} x_{t+1} = {} & f(x_t, K[g(x_t) + \varepsilon\varphi(x_t)\eta_t - g(\overline{x})]) \\ & + \varepsilon\sigma(x_t)\xi_t. \end{aligned} \quad (4)$$

For the asymptotics $z_t = \lim_{\varepsilon\to 0}((x_t^\varepsilon - \overline{x})/\varepsilon)$ of the deviations of solutions x_t^ε of system (4) from the equilibrium $\overline{x}$, the following stochastic system can be written:

$$z_{t+1} = (F + BKC)z_t + BK\varphi\eta_t + \sigma\xi_t, \quad (5)$$

where

$$\begin{aligned} F &= \frac{\partial f}{\partial x}(\overline{x}, 0), \\ B &= \frac{\partial f}{\partial u}(\overline{x}, 0), \\ C &= \frac{\partial g}{\partial x}(\overline{x}), \\ \varphi &= \varphi(\overline{x}), \\ \sigma &= \sigma(\overline{x}). \end{aligned} \quad (6)$$

Due to the uncorrelatedness of random terms η_t and ξ_t, the second moments matrix $M_t = \mathrm{E}(z_t z_t^\top)$ is governed by the equation

$$M_{t+1} = (F + BKC)M_t(F + BKC)^\top + BK\Phi K^\top B^\top + S, \quad (7)$$

where $\Phi = \varphi\varphi^\top$, $S = \sigma\sigma^\top$. A set of matrices K that provide an exponential stability to the equilibrium $\overline{x}$ of the closed deterministic system (4) (with $\varepsilon = 0$ therein) has the following form:

$$\mathbf{K} = \{K \mid \rho(F + BKC) < 1\}, \quad (8)$$

where $\rho(A)$ is a spectral radius of the matrix A. We suppose that the set $\mathbf{K}$ is not empty.

For any $K \in \mathbf{K}$, (7) has a unique stable stationary solution W satisfying the equation

$$W = (F + BKC)W(F + BKC)^\top + BK\Phi K^\top B^\top + S. \quad (9)$$

This matrix W is called the stochastic sensitivity matrix of the equilibrium $\overline{x}$ for system (4). The stochastic sensitivity matrix W approximates a limit behavior of the second moments $\mathrm{E}(x_t^\varepsilon - \overline{x})(x_t^\varepsilon - \overline{x})^\top$ for deviations of solutions x_t^ε from $\overline{x}$:

$$\lim_{t\to\infty} \mathrm{E}(x_t^\varepsilon - \overline{x})(x_t^\varepsilon - \overline{x})^\top \approx \varepsilon^2 W. \quad (10)$$

So, the matrix W characterizes a dispersion of the stationary distributed random states of system (4) around the equilibrium $\overline{x}$.

For any $K \in \mathbf{K}$, the regulator (3) forms a corresponding stochastic equilibrium of system (4) with the stochastic sensitivity matrix W_K which is a solution of (9).

Consider further the following inverse problem.

Problem of Stochastic Sensitivity Synthesis. Let $\mathbf{M}$ be a set of symmetric and positive-definite $n\times n$-matrices. Let $W \in \mathbf{M}$ be some assigned matrix. The problem is to find a feedback matrix $K \in \mathbf{K}$ of regulator (3) such that the equality $W_K = W$ holds. Here, W_K is a solution of (9).

In some cases, this problem can be unsolvable. Therefore, we consider an important notion of the attainability.

Definition 1. An element $W \in \mathbf{M}$ is said to be attainable for system (4) if the equality $W_K = W$ holds for some $K \in \mathbf{K}$.

Definition 2. The set of all attainable elements,

$$\mathbf{W} = \{W \in \mathbf{M} \mid \exists K \in \mathbf{K}, W_K = W\}, \quad (11)$$

is called the attainability set for system (4).

As it follows from (9), the attainability analysis is reduced to the study of solvability of the quadratic matrix equation:

$$\begin{aligned} & BK(CWC^\top + \Phi)K^\top B^\top + BKCWF^\top + FWC^\top K^\top B^\top \\ & \quad + FWF^\top + S - W = 0. \end{aligned} \quad (12)$$

Rewrite (12) with respect to a new unknown matrix

$$L = BK \quad (13)$$

in the following form:

$$L\left(CWC^{\top}+\Phi\right)L^{\top}+LCWF^{\top}+FWC^{\top}L^{\top}+FWF^{\top}+S-W=0. \tag{14}$$

Denote $G(W)=(CWC^{\top}+\Phi)^{1/2}$. Suppose that the matrix $G(W)$ is positive-definite ($G(W)\succ 0$). A substitution $N=LG(W)$ transforms (14) into the following equation:

$$\left(N+F_1\right)\left(N+F_1\right)^{\top}=R\left(W\right), \tag{15}$$

where

$$F_1=FWC^{\top}G^{-1}\left(W\right),$$
$$R\left(W\right)=FWC^{\top}\left(CWC^{\top}+\Phi\right)^{-1}CWF^{\top}-FWF^{\top}-S+W. \tag{16}$$

A necessary condition of (15) solvability is in the nonnegative definiteness of the matrix $R(W)$:

$$R\left(W\right)=FWC^{\top}\left(CWC^{\top}+\Phi\right)^{-1}CWF^{\top}-FWF^{\top}-S+W\succeq 0. \tag{17}$$

Let condition (17) be fulfilled. Then, quadratic equation (15) is equivalent to the linear equation

$$N+F_1=R^{1/2}\left(W\right)J, \tag{18}$$

where J is an arbitrary orthogonal $n\times n$-matrix. It follows from (18) that the feedback matrix K of the regulator (3) which synthesizes the stochastic sensitivity matrix W, satisfies to the linear matrix equation

$$BK=\left(R^{1/2}\left(W\right)J-FWC^{\top}G^{-1}\left(W\right)\right)G^{-1}\left(W\right). \tag{19}$$

In the following theorem, we summarize our theoretical results.

Theorem 3. *Let noises in system (1) and observations (2) be nonsingular ($S\succ 0$, $\Phi\succ 0$).*

(a) If the matrix B is quadratic and nonsingular (rank$B=n=l$) *then*

$$\mathbf{W}=\left\{W\in\mathbf{M}\mid R\left(W\right)\succeq 0\right\}, \tag{20}$$

and, for any matrix $W\in\mathbf{W}$, (19) has a solution

$$K=B^{-1}\left(R^{1/2}\left(W\right)J-FWC^{\top}G^{-1}\left(W\right)\right)G^{-1}\left(W\right)\in\mathbf{K}; \tag{21}$$

(b) If rank$(B)<n$ *then*

$$\mathbf{W}=\left\{W\in\mathbf{M}\mid R\left(W\right)\succeq 0,\ P\left(R^{1/2}\left(W\right)J-FWC^{\top}G^{-1}\left(W\right)\right)=0\right\}, \tag{22}$$

and, for any matrix $W\in\mathbf{W}$, (19) has a solution

$$K=B^{+}\left(R^{1/2}\left(W\right)J-FWC^{\top}G^{-1}\left(W\right)\right)G^{-1}\left(W\right)\in\mathbf{K}. \tag{23}$$

Here, J is an arbitrary orthogonal $n\times n$-matrix, $P=I-BB^{+}$ is a projective matrix, and a "+" sign means a pseudoinversion [23].

3. Controlling of One-Dimensional Stochastic System

Consider one-dimensional discrete stochastic controlled system

$$x_{t+1}=f\left(x_t\right)+u+\varepsilon\sigma\xi_t \tag{24}$$

with noisy observations

$$y_t=x_t+\varepsilon\varphi\eta_t. \tag{25}$$

Here, x, y, u are scalar variables of state, output, and control input; ξ_t, η_t are uncorrelated random scalar sequences with parameters $\mathrm{E}\xi_t=0$, $\mathrm{E}\xi_t^2=1$, $\mathrm{E}\eta_t=0$, and $\mathrm{E}\eta_t^2=1$, and ε, σ, φ are scalar parameters of noise intensities.

It is supposed that the corresponding deterministic uncontrolled system (24) with $\varepsilon=0$ and $u=0$ therein has an equilibrium $\bar{x}:\bar{x}=f(\bar{x})$. In what follows, we use the regulator

$$u_t=k\left(y_t-\bar{x}\right). \tag{26}$$

For the synthesis of the assigned scalar stochastic sensitivity w of the equilibrium $\bar{x}$, we apply theoretical results presented above.

At first describe the attainability set for the considered example. The function $R(W)$ from (17) has here the following representation:

$$R\left(w\right)=\frac{a^2w^2}{w+\varphi^2}-a^2w-\sigma^2+w=\frac{w^2-\left(\left(a^2-1\right)\varphi^2+\sigma^2\right)w-\sigma^2\varphi^2}{w+\varphi^2}, \tag{27}$$
$$a=f'\left(\bar{x}\right).$$

The attainability condition $R(w)\geq 0$ (see Theorem 3 in Section 2) is equivalent to quadratic inequality

$$w^2-\left(\left(a^2-1\right)\varphi^2+\sigma^2\right)w-\sigma^2\varphi^2\geq 0. \tag{28}$$

Thus, all attainable values of w have to satisfy the inequality

$$w\geq w_*=\frac{1}{2}\left[\left(a^2-1\right)\varphi^2+\sigma^2+\sqrt{\left(\left(a^2-1\right)\varphi^2+\sigma^2\right)^2+4\sigma^2\varphi^2}\right]. \tag{29}$$

Note that the value w_* is a minimal value of the stochastic sensitivity that we can provide by this regulator.

Our regulator will synthesize any assign stochastic sensitivity $w\geq w_*$ if we will take (see Theorem 3 in Section 2) a feedback coefficient as follows:

$$k=\frac{\sqrt{w^2-\left(\left(a^2-1\right)\varphi^2+\sigma^2\right)w-\sigma^2\varphi^2}-aw}{w+\varphi^2}. \tag{30}$$

Note that the optimal regulator synthesizing the minimal value of the stochastic sensitivity w_* has the feedback coefficient

$$k_*=-\frac{aw_*}{w_*+\varphi^2}. \tag{31}$$

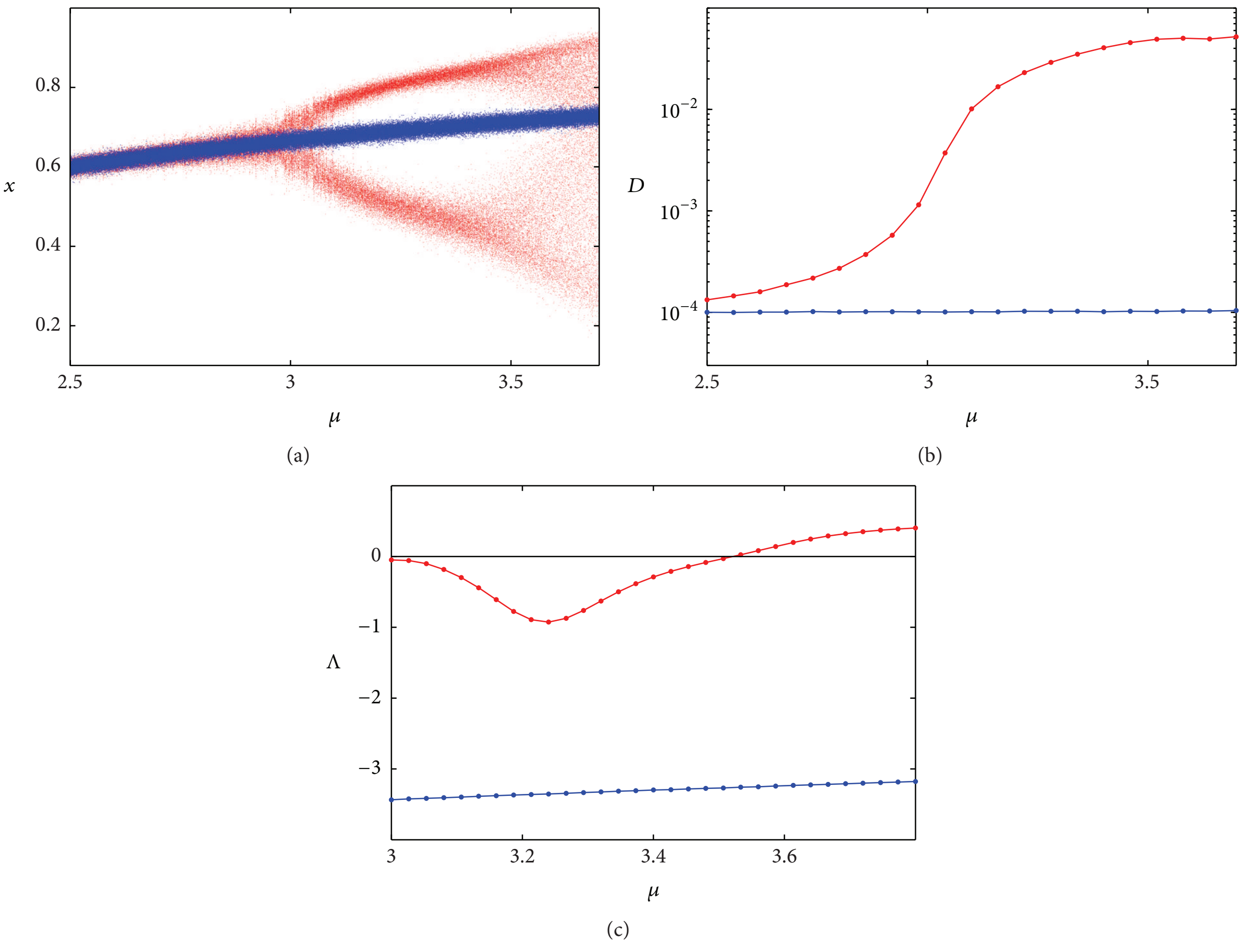

FIGURE 1: Stochastic Verhulst model for $\varepsilon = 0.01$, $\sigma = 1$, and $\varphi = 0.1$ without control (red) and with optimal control (blue): (a) random states; (b) dispersion; (c) largest Lyapunov exponent.

4. Example: Controlling Stochastic Verhulst System

Consider stochastically forced well-known Verhulst system with control and noisy observations:

$$\begin{aligned} x_{t+1} &= \mu x_t \left(1 - x_t\right) + u + \varepsilon \sigma \xi_t, \\ y_t &= x_t + \varepsilon \varphi \eta_t. \end{aligned} \tag{32}$$

The corresponding deterministic uncontrolled system (32) has a nontrivial equilibrium $\overline{x} = 1 - 1/\mu$. This equilibrium is stable for $1 < \mu < 3$ and unstable for $3 < \mu < 4$.

At first consider the influence of random disturbances for system (32) without control ($u = 0$). Under the stochastic disturbances, for random states of this system some probabilistic distribution is formed [24]. In Figure 1(a), random states of system (32), with $u = 0$ for $\varepsilon = 0.01$, $\sigma = 1$ calculated by direct numerical simulation, are plotted by red color.

As the parameter μ crosses the bifurcation value $\mu = 3$ from the left to right, a dispersion of these random states sharply increases. Let $D(\mu)$ be a mean square deviation of random states from the equilibrium $\overline{x}(\mu)$. This function is plotted in Figure 1(b) by red color.

Consider now abilities of control $u_t = k(y_t - \overline{x})$. The aim of the control is to stabilize the equilibrium $\overline{x}(\mu)$ and provide a small dispersion of random states in a wide range of the parameter μ. For the solution of this problem, we will use the optimal regulator that minimizes a stochastic sensitivity of the equilibrium $\overline{x}(\mu)$.

For the Verhulst system, $a(\mu) = 2 - \mu$, and the minimal value w_* of the stochastic sensitivity (see (29)) that we can provide by the regulator has the following explicit parametric representation:

$$\begin{aligned} w_* (\mu) = \frac{1}{2} \Big[& \left(\mu^2 - 4\mu + 3\right) \varphi^2 + \sigma^2 \\ & + \sqrt{\left(\left(\mu^2 - 4\mu + 3\right) \varphi^2 + \sigma^2\right)^2 + 4\sigma^2 \varphi^2} \Big]. \end{aligned} \tag{33}$$

For the optimal regulator that synthesizes this minimal value $w_*(\mu)$, the feedback coefficient

$$k_* = \frac{(\mu - 2)\, w_* (\mu)}{w_* (\mu) + \varphi^2}. \tag{34}$$

Results of the control by this regulator are shown in Figure 1 by blue color. As one can see in Figure 1(a), random states of the controlled system are well localized near the equilibrium $\overline{x}(\mu)$ regardless of whether this equilibrium is stable or not. This optimal control provides the smaller and uniform dispersion (see Figure 1(b)).

Along with the spatial description of the stochastic Verhulst system, consider dynamics of the mutual arrangement of random trajectories in stochastic flows. For quantitative description of this dynamics, the largest Lyapunov exponent Λ is traditionally used [25]. In Figure 1(c), plots of $\Lambda(\mu)$ are shown for uncontrolled (red) and controlled (blue) Verhulst system. A change of $\Lambda(\mu)$ sign from minus to plus justifies a transition of the uncontrolled Verhulst system from order to chaos.

As one can see, the optimal regulator essentially decreases values of $\Lambda(\mu)$, and provides their negativeness for the whole zone of the parameter μ. So, our regulator suppresses chaos and provides a structural stabilization of the system.

5. Conclusion

In this paper, a problem of the stabilization of random states for the nonlinear discrete-time stochastic system with incomplete information was studied. To solve this problem, the stochastic sensitivity technique was developed. Mathematically, the considered control problem was reduced to the analysis of the corresponding quadratic matrix equation. For this equation, we have analyzed its solvability and suggested a constructive method of the solution. A description of attainability sets and algorithms for regulators design were accumulated in Theorem 3. A constructiveness of these general theoretical results was demonstrated for the stochastically forced Verhulst model. It is shown that our method suppresses unwanted large-amplitude oscillations around the equilibria and transforms the system dynamics from chaotic to regular. It is worth noting that the elaborated technique is readily applicable to the control of higher dimensional discrete-time stochastic models.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] J. T. Sandefur, *Discrete Dynamical Modeling*, Oxford University Press, Oxford, UK, 1993.

[2] S. Elaydi, "Is the world evolving discretely?" *Advances in Applied Mathematics*, vol. 31, no. 1, pp. 1–9, 2003.

[3] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Westview Press, 2003.

[4] H. G. Schuster, *Deterministic Chaos*, Physik, Weinheim, Germany, 1984.

[5] B. Ibarz, J. M. Casado, and M. A. F. Sanjuán, "Map-based models in neuronal dynamics," *Physics Reports*, vol. 501, no. 1-2, pp. 1–74, 2011.

[6] W. Horsthemke and R. Lefever, *Noise-Induced Transitions*, Springer Series in Synergetics, Springer, Berlin, Germany, 1984.

[7] K. Matsumoto and I. Tsuda, "Noise-induced order," *Journal of Statistical Physics*, vol. 31, no. 1, pp. 87–106, 1983.

[8] F. Gassmann, "Noise-induced chaos-order transitions," *Physical Review E*, vol. 55, article 2215, 1997.

[9] J. B. Gao, S. K. Hwang, and J. M. Liu, "When can noise induce chaos?" *Physical Review Letters*, vol. 82, no. 6, pp. 1132–1135, 1999.

[10] N. Hritonenko, A. Rodkina, and Y. Yatsenko, "Stability analysis of stochastic Ricker population model," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 64590, 13 pages, 2006.

[11] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Physical Review Letters*, vol. 64, no. 11, pp. 1196–1199, 1990.

[12] A. L. Fradkov and A. Y. Pogromsky, *Introduction to Control of Oscillations and Chaos*, vol. 35, World Scientific, River Edge, NJ, USA, 1998.

[13] G. Chen and X. Yu, *Chaos Control: Theory and Applications*, Springer, New York, NY, USA, 2003.

[14] M. Sanjuan and C. Grebogi, *Recent Progress in Controlling Chaos*, World Scientific, Singapore, 2010.

[15] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics*, Springer-Verlag, New York, NY, USA, 1994.

[16] I. Bashkirtseva, L. Ryashko, and I. Tsvetkov, "Sensitivity analysis of stochastic equilibria and cycles for the discrete dynamic systems," *Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis*, vol. 17, no. 4, pp. 501–515, 2010.

[17] I. Bashkirtseva and L. Ryashko, "Stochastic sensitivity analysis of noise-induced intermittency and transition to chaos in one—dimensional discrete-time systems," *Physica A*, vol. 392, no. 2, pp. 295–306, 2013.

[18] I. Bashkirtseva, "Stochastic phenomena in one-dimensional Rulkov model of neuronal dynamics," *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 495417, 7 pages, 2015.

[19] I. Bashkirtseva and L. Ryashko, "Control of equilibria for nonlinear stochastic discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2162–2166, 2011.

[20] K. J. Astrom, *Introduction to the Stochastic Control Theory*, Academic Press, New York, NY, USA, 1970.

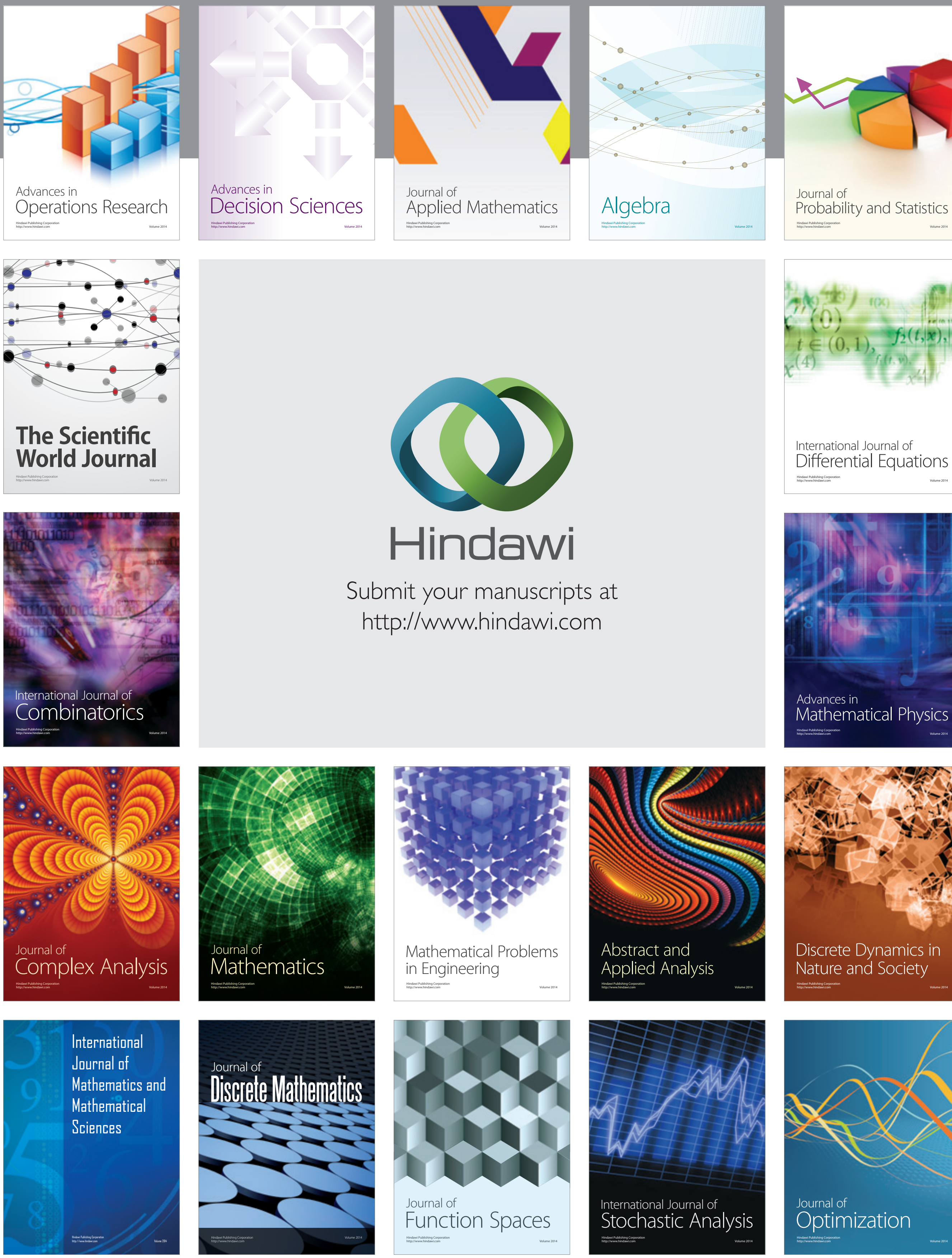
[21] B. Shen, Z. Wang, and H. Shu, *Nonlinear Stochastic Systems with Incomplete Information, Filtering and Control*, Springer, 2013.

[22] H. Dong, Z. Wang, and H. Gao, *Filtering, Control and Fault Detection with Randomly Occurring Incomplete Information*, John Wiley & Sons, Ltd, New York, NY, USA, 2013.

[23] A. Albert, *Regression and the Moore-Penrose Pseudoinverse*, Academic Press, New York, NY, USA, 1972.

[24] J. P. Crutchfield, J. D. Farmer, and B. A. Huberman, "Fluctuations and simple chaotic dynamics," *Physics Reports A: Review Section of Physics Letters*, vol. 92, no. 2, pp. 45–82, 1982.

[25] O. Martin, "Lyapunov exponents of stochastic dynamical systems," *Journal of Statistical Physics*, vol. 41, no. 1-2, pp. 249–261, 1985.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